Non-Markovian Stochastic Liouville equation and its Markovian representation. Extensions of the continuous time random walk approach.

A. I. Shushin
Institute of Chemical Physics, Russian Academy of Sciences,
117977, GSP-1, Kosygin str. 4, Moscow, Russia

Some specific features and extensions of the continuous time random walk (CTRW) approach are analyzed in detail within the Markovian representation (MR) and CTRW-based non-Markovian stochastic Liouville equation (SLE). In the MR CTRW processes are represented by multidimensional Markovian ones. In this representation the probability distribution function (PDF) $W(t)$ of fluctuation renewals is associated with that of reoccurrences in a certain jump state of some Markovian controlling process. Within the MR the non-Markovian SLE, which describes the effect of CTRW-like noise on relaxation of dynamic and stochastic systems, is generalized to take into account the influence of relaxing systems on statistical properties of noise. The generalized non-Markovian SLE is applied to study two modifications of the CTRW approach. One of them considers the cascaded CTRWs in which the controlling process is actually CTRW-like one controlled by another CTRW process, controlled in turn by the third one, etc. Within the MR simple expression for the PDF $W(t)$ of total controlling process is obtained in terms of Markovian variants of controlling PDFs in the cascade. The expression is shown to be especially simple and instructive in the case of anomalous processes determined by long time tailed $W(t)$. The cascaded CTRWs can model the effect of complexity of a system on relaxation kinetics (in glasses, fractals, branching media, ultrametric structures, etc.). Another CTRW-modification describes the kinetics of processes governed by fluctuating $W(t)$. Within the MR the problem is analyzed in a general form without restrictive assumptions on correlations of PDFs of consecutive renewals. The analysis shows that $W(t)$ can strongly affect the kinetics of the process. Possible manifestations of this effect are discussed.

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I. INTRODUCTION

Relaxation and transport phenomena in condensed media are of great importance for understanding of various processes in physics, chemistry, biology etc. [1, 2, 3]. For description of these phenomena a large number of approaches have been proposed in literature [1, 2, 4, 6]. The most popular are Markovian approaches based on Langevin and Fokker-Plank [1, 2] equations assuming negligibly short memory in processes under study. Great advantage of these approaches consists in possibility of relatively simple treatment of relaxation in dynamical and (Markovian) stochastic systems induced by interactions whose fluctuations are Markovian processes (MPs). This type of relaxation, which in what follows for brevity is called MP affected MPs, is known to be described by the stochastic Liouville equation (SLE) [5].

Recent investigations show, however, that Markovian theories do not properly describe the kinetics of relaxations and fluctuations of various observables in many non-equilibrium processes [6, 7]. The non-Markovian relaxation kinetics in these processes results form strong memory effects. Moreover, in a number of systems the kinetics is anomalously long time tailed which implies long time memory in them. The anomaly manifests itself in some interesting phenomena such as non-ergodicity [7, 8, 9], aging [10, 11, 12], slow relaxation [7, 13, 14, 15], etc.

Anomalous specific features of relaxation in highly non-equilibrium systems attract close attentions of increasing number of scientists [7, 13, 14]. Analysis of these features can be made within different models and approximations. One of the most popular and, probably, adequate is continuous time random walk (CTRW) approach in which the process is represented by jump-like fluctuations (or relaxation) treated as a sequence of renewals [4, 7, 16] characterized by the probability density function (PDF) $W(t)$ of waiting times of renewals [this PDF is often denoted as $\psi(t)$ [7, 16, 17]].

Last years different variants have been analyzed and applied to treating a large number of processes [7, 13]. In some recent studies special attention have been paid to above-mentioned anomalously long time tailed relaxation [7, 14]: within the CTRW approach such behavior is assumed to result from that of the PDF $W(t) \sim 1/t^{1+\alpha}$ with $\alpha < 1$ [7, 18]. The applications of the CTRW approach allowed one to interpret a lot of experimental results [7, 14]. It is shown, in particular, that in many cases the specific properties of fluctuations and relaxation can quite properly be described within the anomalous CTRW approach (with anomalous behavior of the PDF $W(t)$).

The CTRW approach is fairly popular last years. It is successfully applied both to general investigations and to interpretation of experimental results. It is worth pointing out, however, that in majority of most interesting applications the validity of this approach is not quite evident. Typically, it is justified with the use of intuitive arguments concerning statistical properties of systems under study. In addition, some recent investigations show that, despite great flexibility, the conventional CTRW

approach proves to be not quite accurate in description of the kinetics of a number of non-Markovian processes [12, 19]. In such a case any further extensions of this approach are of course desirable.

Certain modifications of the conventional CTRW approach have already been discussed in literature. In some CTRW-variants the non-homogeneity of the process, consisting in the dependence of the jump-like fluctuations on the fluctuation number, have been taken into account [20, 21]. In other variants the modification of simple CTRW time sequences of renewals is proposed [22]. These modifications are very interesting and essentially clarify specific features of CTRW-like processes.

In this work we will propose and analyze some extensions of the CTRW approach within the Markovian representation [20, 23, 24]. This method is based on idea that under fairly general assumptions a wide variety of non-Markovian CTRW processes can be treated as MPs with fluctuating parameters, whose fluctuations are described by other MPs called hereafter controlling processes. In so doing these CTRW processes are unambiguously represented by some multidimensional MPs [20] and the statistics of renewals is associated with that of reoccurrences in certain transition state (or the state of onset of fluctuation jumps) during the controlling MPs in some auxiliary spaces. In particular, the PDF $W(t)$ is expressed in terms of the characteristic functions of the controlling processes.

The Markovian representation is shown to be very useful for the analysis of non-Markovian CTRW-like models describing different physical processes [20, 23, 24]. In addition it permits rigorous derivation of the non-Markovian stochastic Liouville equation (SLE) treating relaxation in dynamical and Markovian stochastic systems induced by CTRW-type fluctuating interactions with thermal bath [20, 23, 24]. In what follows, for brevity, Markovian systems in which relaxation is induced by CTRW-fluctuating interactions will be called CTRW affected MPs.

In this work the compact formulation of the Markovian representation is proposed which simplifies and generalizes derivation of the CTRW-based non-Markovian SLE thus allowing for extension of the SLE to take into account back effect of fluctuating system on effective statistical properties of fluctuations. This formulation especially clearly demonstrates that CTRW processes can be considered as MP affected MPs (in above-proposed brief terminology) and basic equations of the CTRW approach are nothing else but the SLEs in a reduced form.

The obtained generalized non-Markovian SLE is applied to the analysis of validity of CTRW approaches. Two important extensions of the CTRW approach are proposed and discussed in detail:

The first modification describes the effect of cascaded controlling processes in which the process controlling renewals is assumed to be CTRW-like one controlled by the second CTRW process, which in turn is controlled by the third CTRW processes, etc. In this cascaded CTRW control model the compact representation for the PDF $W(t)$ (more correctly for the Laplace transform of this function) is obtained in terms of PDFs of controlling processes $W_j(t)$ at all cascade steps j, found assuming these processes to be Markovian (i.e. neglecting control). This model is very suitable for the analysis of non-Markovian relaxation kinetics in structured and disordered systems especially in the case of anomalous long time tailed behavior of PDFs $W_j(t)$.

The second modification treats CTRW-like processes governed by fluctuating PDFs $W(t)$. This type of processes can be considered as an extension of conventional CTRW-approach in which fluctuations of $W(t)$ are assumed to result from the additional effect of non-equilibrium medium and are modeled by the dependence of the system on the Markovian stochastic variable of special type. In this model and within the Markovian representation the description of these processes reduces to solving the non-Markovian SLE. The analysis of the model shows that fluctuations of the PDF $W(t)$ can strongly change the kinetics of CTRW relaxation. Especially significant effect is expected in the case of anomalous long time tailed processes.

II. GENERAL FORMULATION

We consider relaxation processes in a dynamical or stochastic Markovian system induced by fluctuating interaction with the classical thermal bath. The Markovian evolution of the system is assumed to be governed by the fluctuating Liouville operator $L(t)$. Our general formulation is applicable both to quantum and classical systems. In particular, in the dynamic systems the only difference of these two cases is in the form of the operators L: for a quantum system $L = i[H, \dots]$, (here $[H, \dots]$ is the commutator with the Hamiltonian H of the system), while for a classical system $L = \{H, \dots\}$ (with $\{H, \dots\}$ being the classical Poisson brackets). In what follows, for definiteness and brevity, we will concentrate on general results as applied to classical systems.

Fluctuations of $L(t)$ are assumed to result from the dependence on the fluctuating bath coordinate $\mathbf{x}$: $L(t) \equiv L_{\mathbf{x}(t)}$, whose changes are modeled by stochastic jumps between states $|\nu\rangle \equiv |\mathbf{x}_\nu\rangle$ with different $L = L_\nu$ in the multidimensional space $\{x\}$.

Hereafter we will use "bra-ket" notation for the states in $\{x\}$-space suitable for treating relaxation phenomena determined by not self-adjoint evolution operators.

In the model under study the system evolution is described by the (PDF) $\rho(t)$ (or density matrix for quantum systems) which satisfies the linear equation

$$\dot{\rho} = -\hat{L}\rho \ \text{ with } \ \hat{L} \equiv \hat{L}_{\mathbf{x}} = \sum\nolimits_\nu |\mathbf{x}_\nu\rangle L_\nu \langle \mathbf{x}_\nu|. \tag{2.1}$$

For simplicity of presentation, $\hat{L}$ is considered to be diagonal in $\{x\}$-space though the formulas obtained are valid in the case non-diagonal $\hat{L}$ as well.

The operator solution of eq. (2.1) is represented as

$$\rho(t) = \hat{U}(t)\rho_0 \text{ with } \hat{U}(t) = \hat{T}\Big[e^{-\int_0^t d\tau \hat{L}(\tau)}\Big], \qquad (2.2)$$

where $\hat{T}$ is the time ordering operator and $\rho_0 = \rho(t=0)$ is the initial condition.

Experimentally measured observables are usually described by the evolution operator $\hat{U}(t)$ averaged over stochastic fluctuations of $\mathbf{x}(t)$ which is expressed in terms of the conditional evolution operator $\hat{\mathbb{G}}(x, x'|t)$ as

$$\langle \hat{U}(t)\rangle_{\mathbf{x}} = \sum\nolimits_{\mathbf{x},\mathbf{x}_0} \hat{\mathbb{G}}(\mathbf{x}, \mathbf{x}_0|t)\sigma_i(\mathbf{x}_0) \equiv \langle \mathbf{x}_e|\hat{\mathbb{G}}|\mathbf{x}_i\rangle, \qquad (2.3)$$

where $\sigma_i(\mathbf{x}) \equiv |\mathbf{x}_i\rangle$ is the initial PDF of the system in $\{x\}$-space normalized by the condition $\sum_{\mathbf{x}} \sigma_i(\mathbf{x}) = 1$. In eq. (2.3) we have also introduced the (adjoined) equilibrium state vector which in bra-ket notation is represented as [see below eq. (3.8)]: $\langle \mathbf{x}_e| = \sum_\nu \langle \mathbf{x}_\nu|$.

In accordance with eq. (2.3) the problem reduces to evaluating the operator $\hat{\mathbb{G}}(\mathbf{x}, \mathbf{x}_i|t)$. Unfortunately this can be done relatively easily only for very few models of stochastic $\mathbf{x}(t)$-fluctuations. In this work we will analyze some of those models based on the CTRW-approach.

III. MODELS OF FLUCTUATIONS

A. Markovian models

The Markovian approach is based on the assumption that $\hat{L}(t)$-fluctuations are described by the stochastic MPs in $\{x\}$-space and $\hat{L}(t)$-evolution is described by the PDF $\sigma(\mathbf{x}, \mathbf{x}_0|t)$ satisfying equation

$$\dot{\sigma} = -\hat{\mathcal{L}}\sigma \quad \text{with} \quad \sigma(\mathbf{x}, \mathbf{x}_0|0) = \delta_{\mathbf{x},\mathbf{x}_0}, \qquad (3.1)$$

where $\hat{\mathcal{L}} \equiv \hat{\mathcal{L}}_{\mathbf{x}}$ is some linear operator in $\{x\}$-space. The principal simplification of the problem results from the fact that in the Markovian approach (3.1), i.e. in the case of MP affected MPs, $\hat{\mathbb{G}}(\mathbf{x}, \mathbf{x}_0|t)$ obeys the SLE:

$$\dot{\hat{\mathbb{G}}} = -(\hat{L} + \hat{\mathcal{L}})\hat{\mathbb{G}}, \quad \text{so that} \quad \hat{\mathbb{G}}(t) = e^{-(\hat{L}+\hat{\mathcal{L}})t}. \qquad (3.2)$$

which yields for $\langle \tilde{\hat{U}}\rangle = \int_0^\infty dt\, \langle \hat{U}(t)\rangle \exp(-\epsilon t)$:

$$\langle \tilde{\hat{U}}\rangle = \langle \tilde{\hat{\mathbb{G}}}\rangle_{\mathbf{x}} = \langle \mathbf{x}_e|(\epsilon + \hat{L} + \hat{\mathcal{L}})^{-1}|\mathbf{x}_i\rangle. \qquad (3.3)$$

In the form (3.2) the SLE is valid for any dependence $\hat{L}_{\mathbf{x}}$ on the coordinate $\mathbf{x}$ though, in general, it is still very complicated for analysis either numerical or analytical.

Significant simplification can be gained within some special models, for example, in the CTRW approach.

B. CTRW-based models

Non-Markovian $\hat{L}(t)$-fluctuations can conveniently be described by the CTRW approach [4, 7, 17]. It treats fluctuations as a sequence of sudden, jump-like changes of $\hat{L}$. In the simplest variant of the CTRW (more complicated variants are discussed below) the onset of any particular change of number β is described by the probability $P_{\beta-1}(t)$ (in $\{x\}$-space) not to have any change during time t and its derivative $W_{\beta-1}(t) = -\dot{P}_{\beta-1}(t)$, i.e. the PDF for times of waiting for the change. These functions are independent of β for $\beta > 1$ and for $\beta = 1$ depend on the problem considered [4, 7]:

$$W_{\beta\geq 1}(t) = W_n(t), \quad W_0(t) = W_i(t), \qquad (3.4)$$

and $P_j(t) = \int_0^t d\tau\, W_j(\tau)$, $(j = n, i)$,. In what follows we will mainly discuss the non-stationary CTRW variant in which $W_{\beta\geq 0}(t) = W_n(t)$.

The Laplace transforms of $W_j(t)$ and $P_j(t)$, $(j = n, i)$), are related by simple equation $\widetilde{P_j}(\epsilon) = [1 - \widetilde{W_j}(\epsilon)]/\epsilon$ with

$$\widetilde{W_j}(\epsilon) = [1 + \Phi_j(\epsilon)]^{-1}, \ (j = n, i). \qquad (3.5)$$

In eq. (3.5) $\Phi_j(\epsilon)$ is the important auxiliary function [20].

1. *Markovian representation*

The results of our earlier studies [20, 23] show that important non-Markovian generalizations of the SLE (3.2) can be obtained by assuming the operator $\hat{\mathcal{L}}(t)$ to be a stochastic function of time.

In this work we will analyze quite natural generalization of the Markovian SLE (3.2) based on the assumption that fluctuations of $\hat{\mathcal{L}}(t)$ are Markovian. In other words $\hat{\mathcal{L}}(t) \equiv \hat{\mathcal{L}}_{\mathbf{z}(t)}$ is assumed to be a function of the Markovian stochastic (controlling) variable $\mathbf{z}(t)$ which, in general, is a vector, so that generalizing the solution (3.2) of the Markovian SLE one can write

$$\hat{\mathbb{G}}(t) = \hat{T}\{\exp[-\textstyle\int_0^t d\tau(\hat{L}_{\mathbf{x}(\tau)} + \hat{\mathcal{L}}_{\mathbf{z}(\tau)})]\}. \qquad (3.6)$$

In the major part of the analysis we will assume that both $\{x\}$- and $\{z\}$-spaces are discrete and use bra/ket notation with Greek and Latin characters for states in these two spaces, for example, $|\mathbf{x}_\nu\rangle$ and $|\mathbf{z}_j\rangle$, respectively (although, in case of need the continuous variants of $\{x\}$- and $\{z\}$-spaces will also be considered).

We start our analysis with the simple Markovian stochastic migration process in $\{x\}$-space, which is described by the Kolmogov-Feller jump matrix

$$\hat{\mathcal{L}} = \hat{k}^d_{\mathbf{z}(t)} - \hat{P}_x \hat{k}^n_{\mathbf{z}(t)} \text{ with } \hat{P}_x = \textstyle\sum_{\nu\neq\mu} p_{\nu\mu}|\mathbf{x}_\nu\rangle\langle \mathbf{x}_\mu| \qquad (3.7)$$

in which $\hat{k}^d_{\mathbf{z}(t)} \equiv \hat{k}^d[\mathbf{z}(t)]$ and $\hat{k}^n_{\mathbf{z}(t)} \equiv \hat{k}^n[\mathbf{z}(t)]$ are the matrices (diagonal in $\{x\}$-space) of z-dependent, i.e. fluctuating in time, jump rates and $p_{\nu\mu}$ are the probabilities of jumps $\{x\}$-space normalized by the relation $\sum_\nu p_{\nu\mu} = 1$. For simplicity, we assume that $\hat{P}_x$ is independent of z, though most general results obtained in this section are valid in the case of z-dependent $\hat{P}_x$ as well (see below).

The matrix $\hat{\mathcal{L}}(t)$ describes relaxation in $\{x\}$-space to the equilibrium state

$$|\mathbf{x}_e\rangle = \sum_\nu p^e_\nu|\mathbf{x}_\nu\rangle, \quad \text{with} \quad \langle\mathbf{x}_e| = \sum_\nu\langle\mathbf{x}_\nu|, \tag{3.8}$$

for which $(1-\hat{P}_x)\hat{k}|\mathbf{x}_e\rangle = 0$ and $\langle\mathbf{x}_e|(1-\hat{P}_x)\hat{k} = 0$. This state is assumed to be independent of $\mathbf{z}$. Noteworthy is that even in the absence of the equilibrium state $|\mathbf{x}_e\rangle$, for example when $\hat{\mathcal{L}}$ describes diffusive migration in infinite space, the adjoint vector $\langle\mathbf{x}_e|$ defined in eq. (3.8) still exists and satisfies the relation $\langle\mathbf{x}_e|(1-\hat{P}_x)\hat{k} = 0$ which means nothing else but the conservation of population in the process of migration in $\{x\}$-space.

Within the considered Markovian approximation for $\mathcal{L}_{\mathbf{z}(t)}$-fluctuations the evolution of the system in controlling $\{z\}$-space is governed by the PDF $\varphi(\mathbf{z},\mathbf{z}_0|t)$ satisfying equation

$$\dot{\varphi} = -\hat{\Lambda}\varphi \quad \text{with} \quad \varphi(\mathbf{z}|0) = |\varphi_i(\mathbf{z})\rangle \equiv |\mathbf{z}_i\rangle, \tag{3.9}$$

in which $\hat{\Lambda} \equiv \hat{\Lambda}_{\mathbf{z}}$ is some linear operator describing relaxation in $\{z\}$-space and $\varphi_i(\mathbf{z}) \equiv |\mathbf{z}_i\rangle$ is the initial state of the controlling process (in this $\{z\}$-space) in the par-ket representation. Similar to the operator $\mathcal{L}$ in $\{x\}$-space for our further applications it is convenient to introduce the equilibrium states of the operator $\hat{\Lambda}$

$$|\mathbf{z}_e\rangle = \sum_j p^{\mathbf{z}}_j|\mathbf{z}_j\rangle \quad \text{and} \quad \langle\mathbf{z}_e| = \sum_j\langle\mathbf{z}_j|. \tag{3.10}$$

As in the case of $\{x\}$-space considered above, the vector $\langle\mathbf{z}_e|$ is still meaningful even in the absence of the equilibrium state of the operator $\hat{\Lambda}$.

The operator $\hat{\Lambda}$ can, in general, depend on $\mathbf{x}$. Moreover, $\hat{\Lambda}$ can be a non-diagonal matrix in $\{x\}$-space though, for simplicity of further discussion, we will assume $\hat{\Lambda}$ to be diagonal in this space.

In our analysis, in case of need we will use the simple and flexible continuous Smoluchowski model for controlling process in $\{z\}$-space in which

$$\hat{\Lambda} = -D_z z^{1-n_z} e^{-u(z)}\nabla_z[z^{n_z-1}e^{u(z)}\nabla_z], \tag{3.11}$$

with $z = |\mathbf{z}|$, $u(z)$ is the effective potential, and $\nabla_z = \partial/\partial z$, is the radial part of the operator describing diffusion in n_z-dimensional $\{z\}$-space with the diffusion coefficient D_z. This model is quite sufficient for our qualitative and semiquantitative analysis. The correspondence between continuous model implying smooth functions D_z and $u(z)$ and its discrete variant considered above is formulated as follows: $\mathbf{z} \leftrightarrow b|\mathbf{z}_j\rangle$, where b is the spacing in the discrete model.

Similarly to the simplest model discussed in Sec. III.A one can easily see that in this approximation the evolution operator $\hat{\mathbb{G}}_{\mathbf{r}}(\mathbf{r},\mathbf{r}_i|t)$, which determines $\langle\hat{U}(t)\rangle$ obeys the Markovian SLE in the extended space $\{\mathbf{r}\}$ including the dynamical subspace $\{x\}$ and additional stochastic subspace $\{z\}$: $\{\mathbf{r}\} = \{x \otimes z\}$:

$$\dot{\hat{\mathbb{G}}}_{\mathbf{r}} = -(\hat{L} + \hat{\mathcal{L}} + \hat{\Lambda})\hat{\mathbb{G}}_{\mathbf{r}} \quad \text{with} \quad \hat{\mathbb{G}}_{\mathbf{r}}(\mathbf{r},\mathbf{r}_i|0) = \delta_{\mathbf{r}\mathbf{r}_i}, \tag{3.12}$$

so that we get for $\hat{\mathbb{G}}$ and the Laplace transform $\tilde{\hat{U}} = \langle\tilde{\hat{\mathbb{G}}}\rangle_x$:

$$\hat{\mathbb{G}} = \langle\mathbf{z}_e|\hat{\mathbb{G}}_{\mathbf{r}}|\mathbf{z}_i\rangle \quad \text{and} \quad \langle\tilde{\hat{U}}\rangle = \langle(\hat{\Omega} + \hat{\mathcal{L}} + \hat{\Lambda})^{-1}\rangle_{\mathbf{r}}, \tag{3.13}$$

where $\hat{\Omega} = \epsilon + \hat{L}$. In the form (3.13) the SLE is valid for any dependence of coupling $\hat{\mathcal{L}}_z$ on the coordinate z though, in general, it is still very complicated for analysis either numerical or analytical.

Significant simplification can be gained within some special models for jump-rate matrix $\hat{k}_z$ [see eq. (2.3)]. One of the most well known is the CTRW-model discussed below, in Sec. IVB.

2. *Localized jump rates. Non-Markovian SLE*

Very important results can be obtained in the model of interaction $\hat{\mathcal{L}}_z$ highly localized in $\{z\}$-space, i.e. highly localized jump rates $\hat{k}^d_z$ and $\hat{k}^n_z$. The most interesting for our further discussion is the simplest variant of this model, which in the case of discrete $\{z\}$-space is written as

$$\hat{k}^d_{\mathbf{z}} = \hat{k}_0\hat{P}^{tt}_{\mathbf{z}}, \quad \hat{k}^n_{\mathbf{z}} = \hat{k}_0\hat{P}^{nt}_{\mathbf{z}} \text{ with } \hat{P}^{jt}_{\mathbf{z}} = |\mathbf{z}_j\rangle\langle\mathbf{z}_t| \tag{3.14}$$

and $j = t, n$. Here $\hat{k}_0$ is the matrix diagonal in $\{x\}$-space, while $|\mathbf{z}_n\rangle$ and $|\mathbf{z}_t\rangle$ are the states $\{z\}$-space. The model (3.14) approximates $\mathcal{L}(t)$-fluctuations by the set of jumps (blips) with the relaxation effect of each jump described by $\hat{k}_0$-matrix. As to properties of jumps statistics, they are determined by the controlling operator $\hat{\Lambda}$ and the localized matrix $\hat{P}^{nt}_{\mathbf{z}}$ which implies that jumps occur as long as the system appears at the state $|\mathbf{z}_t\rangle$ and each jump results in the transition $|\mathbf{z}_t\rangle \to |\mathbf{z}_n\rangle$ in $\{z\}$-space.

Evidently, for the case of continuum $\{z\}$-space in eq. (3.14) [with $\hat{\Lambda}$, for example, of type of that given by eq. (3.11)] the term $\hat{P}^{nt}_{\mathbf{z}}$ should be replaced by the corresponding δ-function type one.

Within the model (3.14) general formula (3.13) can be represented in a more suitable CTRW-like form:

$$\begin{aligned}\tilde{\hat{\mathbb{G}}}(\epsilon) &= \langle\mathbf{z}_e|\hat{G}(\epsilon)[1 - \hat{P}_x\hat{k}_0\tilde{\hat{G}}(\epsilon)]^{-1}|\mathbf{z}_i\rangle \\ &= \tilde{\hat{\mathbb{P}}}_i(\epsilon) + \tilde{\hat{\mathbb{P}}}_n(\epsilon)[1 - \tilde{\widehat{\mathbb{W}}}_n(\epsilon)]^{-1}\tilde{\widehat{\mathbb{W}}}_i(\epsilon), \end{aligned} \tag{3.15}$$

where

$$\tilde{\hat{G}}(\epsilon) = (\epsilon + \hat{L} + \hat{k}^d_{\mathbf{z}} + \hat{\Lambda})^{-1}, \tag{3.16}$$

is the Green's function, that essentially determines two matrices: the effective PDF-matrix $\tilde{\widehat{\mathbb{W}}}_j(\epsilon)$ of sudden fluctuation jumps (analogs of renewals) and the matrix $\tilde{\hat{\mathbb{P}}}_j(\epsilon)$ of probabilities not to observe any fluctuation until time t, playing important role in CTRW-based theories [4, 7, 17]] [see eq. (3.4)]. These two matrices are expressed in terms of the Green's function of the controlling process in $\{z\}$-space:

$$\tilde{\widehat{\mathbb{W}}}_j(\epsilon) = \hat{P}_x\tilde{\hat{w}}_j(\epsilon), \ (j = i, n), \tag{3.17}$$

in which

$$\tilde{\hat{w}}_j(\epsilon) = \hat{k}_0 \langle \mathbf{z}_t | \hat{G}(\epsilon) | \mathbf{z}_j \rangle = [1 + \hat{\Phi}_j(\epsilon)]^{-1}, \qquad (3.18)$$

where

$$\hat{\Phi}_j(\epsilon) = [\hat{g}_{tt}(\epsilon)\hat{g}_{tj}^{-1}(\epsilon) - 1] + [\hat{k}_0 \hat{g}_{tj}(\epsilon)]^{-1} \qquad (3.19)$$

with

$$\hat{g}_{tj} = \langle \mathbf{z}_t | \hat{g}(\epsilon) | \mathbf{z}_j \rangle \text{ and } \hat{g}(\epsilon) = (\epsilon + \hat{L} + \hat{\Lambda})^{-1}, \qquad (3.20)$$

and

$$\tilde{\hat{\mathbb{P}}}_j(\epsilon) = \langle \mathbf{z}_e | \hat{G}(\epsilon) | \mathbf{z}_j \rangle = (\epsilon + \hat{L})^{-1}[1 - \tilde{\hat{w}}_j(\epsilon)]. \qquad (3.21)$$

In eq. (3.21) $\langle \mathbf{z}_e |$ is equilibrium ket-vector in $\{z\}$-space (with $\langle \mathbf{z}_e | \Lambda = 0)$ defined by formula similar to eq. (3.8) but with $\mathbf{x}$-vectors replaced by those in $\{z\}$-space.

Expression (3.15), which in what follows will be called the generalized non-Markovian SLE (in resolvent form), looks similar to the simpler one derived earlier within the Markovian representation [20]]. Moreover, for $\hat{L} = 0$ it reduces to that obtained in refs. [20, 25] in which the PDF matrices

$$\tilde{\hat{\mathbb{W}}}_j(\epsilon) = \tilde{\hat{\mathbb{W}}}_{0j}(\epsilon) = \hat{P}_x \tilde{\hat{W}}_j(\epsilon), \qquad (3.22)$$

$$\tilde{\hat{\mathbb{P}}}_j(\epsilon) = \tilde{\hat{\mathbb{P}}}_{0j}(\epsilon) = \epsilon^{-1}[1 - \tilde{\hat{W}}_j(\epsilon)], \quad (j = i, n), \qquad (3.23)$$

where

$$\tilde{\hat{W}}_j(\epsilon) = \int_0^\infty dt \, \langle \mathbf{z}_e | \hat{G}_0(t) | \mathbf{z}_j \rangle e^{-\epsilon t} \equiv \langle \mathbf{z}_e | \tilde{\hat{G}}_0(\epsilon) | \mathbf{z}_j \rangle \qquad (3.24)$$

is the PDF matrix in which $\hat{G}_0(t) = e^{-(\hat{k}^d_{\mathbf{z}} + \hat{\Lambda})t}$ and $\tilde{\hat{G}}_0(\epsilon) = (\epsilon + \hat{k}^d_{\mathbf{z}} + \hat{\Lambda})^{-1}$, are the Green's functions describing the stochastic process in $\{z\}$-space. Equations (3.22) and (3.23) can be considered as a generalized variant of relation (3.5).

Despite the similarity of obtained formulas with those known in the CTRW theory [4, 7], there are, however, some important specific features of the general expression (3.15) as opposed to the conventional CTRW formulas (3.4)-(3.5) and (3.22), (3.23).

1) The matrix

$$\tilde{\hat{\mathbb{P}}}_j(\epsilon) = \hat{\Omega}^{-1}(\epsilon)[1 - \hat{k}_0 \langle \mathbf{z}_t | (\hat{\Omega}(\epsilon) + \hat{k}_0 + \hat{\Lambda})^{-1} | \mathbf{z}_j \rangle] \qquad (3.25)$$

in which $\hat{\Omega}(\epsilon) = \epsilon + \hat{L}$, can be non-diagonal, unlike similar matrix in the conventional CTRW approach (see below). Moreover, in general, elements $\hat{\mathbb{P}}_{j_{ik}}(t)$ of the matrix $\hat{\mathbb{P}}_j(t)$ do not satisfy the evident relation of type of $\hat{\mathbb{P}}_{j_{ik}}(0) = 1$ which would allow one to interpret these elements as probabilities. In addition, $\hat{\mathbb{P}}_j(t)$ can be complex values, for example, in quantum processed for which the elements of $\hat{L}$ are complex.

2) The relation between $\tilde{\hat{\mathbb{W}}}_j(\epsilon)$ and $\tilde{\hat{\mathbb{P}}}_j(\epsilon)$ implied by eqs. (3.17) and (3.21) does not coincide with that between the PDF-matrix of waiting times and the matrix of probabilities known in the CTRW theory. Strictly speaking, the conventional relation [see eqs. (3.22) and (3.23)] is reproduced only in the evident case $\hat{L} = 0$.

3) Equation (3.15) is of the form of CTRW expression however with matrices $\tilde{\hat{\mathbb{W}}}_j(\epsilon)$ and $\tilde{\hat{\mathbb{P}}}_j(\epsilon)$ strongly modified by the dynamic operator $\hat{L}$ [in contrast to the non-Markovian SLE appealing to the conventional PDF matrices $\hat{\mathbb{W}}_{0j}(t)$ and $\hat{\mathbb{P}}_{0j}(t)$ [20] (see eq. (3.26))].

The above-mentioned effects of dynamic evolution can result in significant change of the time dependencies $\hat{\mathbb{W}}_{0j}(t)$ and $\hat{\mathbb{P}}_{0j}(t)$ thus leading to the strong change of the kinetics of relaxation processes under study.

3. CTRW-based non-Markovian SLE.

The simplified variant of formula (3.15) can be obtained in the special case of matrix $\tilde{\hat{W}}_j(\epsilon)$ diagonal in $\{x\}$-space, which is realized when the matrices $\hat{\Lambda}$, $\hat{L}$, and $\hat{k}_0$ commute with each other: $[\hat{\Lambda}, \hat{L}] = [\hat{L}, \hat{k}_0] = 0$. In this case one can obtain the representation for $\tilde{\hat{\mathbb{W}}}_j(\epsilon)$ in terms of the Laplace transform of the conventional PDF-matrix of fluctuation blips $\tilde{\hat{\mathbb{W}}}_{0j}$ [see eqs. (3.22)) and (3.24))]:

$$\tilde{\hat{\mathbb{W}}}_j(\epsilon) = \tilde{\hat{\mathbb{W}}}_{0j}(\hat{\Omega}(\epsilon)) = \hat{P}_x \int_0^\infty dt \, \hat{W}_j(t) e^{-\hat{\Omega}(\epsilon)t}, \qquad (3.26)$$

where $\hat{\Omega}(\epsilon) = \epsilon + \hat{L}$.

The simplified non-Markovian SLE (3.26), is nevertheless more general than the variant of this equation in which $\tilde{\hat{W}}_{0j}(\epsilon)$ independent of x-coordinate (i.e. $\tilde{\hat{W}}_{0j}(\epsilon)$ proportional to the unity matrix in $\{x\}$-space) [26].

Noteworthy is that, in general, if $\hat{\Lambda}$, $\hat{L}$, and $\hat{k}_z$ do not commute with each other, the representation (3.26) is not valid and one should use the original expression (3.15).

IV. EXTENSIONS OF CTRW MODELS

The proposed Markovian representation, based on description of CTRWs as MP affected MPs, enables one to significantly extend the CTRW approach and the non-Markovian SLE. It reduces the treatment of system evolution to averaging exponential functional over Markovian fluctuations and offers the expression of CTRW equations in terms of multidimensional Markovian SLE.

There are some straightforward extensions which do not need detailed analysis. For example, the evident variant of extension is the model of several highly localized jump states in $\{z\}$-space $\langle \mathbf{z}_{t_i} |$ and $| \mathbf{z}_{n_i} \rangle$ ($1 \le i \le i_z$, $i_z > 1$) in which $k_z = \sum_j k_0^{(j)} \hat{P}_{nt}^{(j)}$ with $\hat{P}_{nt}^{(j)} = | \mathbf{z}_n \rangle \langle \mathbf{z}_t |$. In this model the general formula (3.13) is also simplified by reducing the problem to solving the linear equation for the matrix $\hat{\mathbf{G}}$ with elements $\hat{\mathbf{G}}_{ij}(\epsilon) = \langle \mathbf{z}_{t_i} | \hat{G}(\epsilon) | \mathbf{z}_{n_j} \rangle$.

Unfortunately, in this model the expression for $\hat{\tilde{\mathbb{G}}}$ is fairly cumbersome and not quite suitable for applications.

In this section we will mainly discuss less evident and more general extensions which allow for describing effects of Markovian and non-Markovian fluctuating interactions on kinetics of relaxation in CTRW-like systems, i.e. (in our brief terminology) MP and CTRW affected CTRWs. In the Markovian representation the problem is still reduced to the analysis of MP affected MPs though in multidimensional space.

Because of large number of parameters in this type of multidimensional MPs it is practically senseless to discuss the problem in general. For this reason, here we will restrict ourselves to consideration of two modifications and extensions of the CTRW approach interesting for theoretical analysis and for applications.

A. CTRW-control. Coupled Markovian processes.

1. General results

So far in our analysis we have discussed the Markovian model for $\hat{\mathcal{L}}(t)$ fluctuations which suggests that the controlling stochastic process $z(t)$, responsible for fluctuations of the jump rates $\hat{k}^{d,n}_{\mathbf{z}(t)}$ [see eq. (3.7)], is Markovian.

Here we will consider the extension of the Markovian model based on the assumption that the controlling process $z(t)$ is represented as a sequence (cascade) of controlling Markovian processes in the multidimensional space $\{\mathbf{Z}_1\} = \{z_1, z_2, \ldots, z_N\}$ with evolution in each space $\{z_q\}$ being controlled by the process in $\{z_{q+1}\}$-space whose mechanism is similar to that responsible for CTRW-type motion in $\{x\}$-space as described above in Sec. III.B. This mechanism implies that stochastic jumps in $\{z_q\}$-space happen as long as the system appears in the transition state $|\mathbf{z}^t_{q+1}\rangle$ (in $\{z_{q+1}\}$-space). As compared to the mechanism presented in Sec III.B, however, here, for simplicity, we will assume that jumps are not accompanied by the change of state in $\{z_{q+1}\}$-space, i.e. the final state $|\mathbf{z}^n_{q+1}\rangle = |\mathbf{z}^t_{q+1}\rangle$. For the same reason we will also assume that initial state $|\mathbf{z}^i_q\rangle$ of evolution in $\{z_q\}$-subspace coincides with the transition state: $|\mathbf{z}^i_q\rangle = |\mathbf{z}^t_q\rangle$ $(1 \le q \le N)$.

The above-formulated model of cascaded controlling processes is described by the following jump operators in spaces $\{\mathbf{Z}_q\} = \{z_q, z_2, \ldots, z_N\}$ $(q \ge 1)$

$$\hat{\mathcal{L}}_q = \sum\nolimits_{i=q}^{N} \hat{\Lambda}_i \hat{P}^{tt}_{i+1}, \quad \text{where} \quad \hat{\Lambda}_i = (1 - \hat{\mathcal{P}}_{z_i})\hat{k}_{0_i} \tag{4.1}$$

with

$$\hat{P}^{tt}_i = |\mathbf{z}^t_i\rangle\langle\mathbf{z}^t_i|. \tag{4.2}$$

Here $\hat{\mathcal{P}}_{z_i}$ is the matrix of distribution functions of jump lengths (in $\{z_i\}$-space) and $\hat{k}_{0_i}$ is the matrix of jump rates diagonal in $\{z_i\}$-space.

Similar to the model discussed in Sec. III.B, the control of motion in $\{x\}$-space will be described by z-dependence of the operator $\mathcal{L}$:

$$\hat{\mathcal{L}}_{\mathbf{z}} = (1 - \hat{\mathcal{P}}_x)\hat{k}_0 \hat{P}^{tt}_1. \tag{4.3}$$

Noteworthy is that in accordance with obtained results the model (4.1)-(4.3) can be considered as a generalized variant of CTRW-type models for the controlling process.

Recall that the problem under study reduces to evaluating the evolution operator $\hat{\mathbb{G}}(t)$ defined in eq. (3.6). In the proposed model of cascaded controlling processes this operator can be found in analytical form.

Formula for $\hat{\mathbb{G}}(t)$ can be obtained with the use of the general expression (3.15)

$$\begin{aligned} \hat{\tilde{\mathbb{G}}}(\epsilon) &= \epsilon^{-1}[1 - \hat{w}(\epsilon)][1 - \hat{P}_x\hat{w}(\epsilon)]^{-1} && (4.4)\\ &= \epsilon^{-1}\hat{\Phi}(\epsilon)[\hat{\Phi}(\epsilon) + \hat{\mathcal{L}}_x\hat{k}_0^{-1}]^{-1} && (4.5) \end{aligned}$$

in which $\hat{\mathcal{L}}_x = (1 - \hat{P}_x)\hat{k}_0$,

$$\hat{w}(\epsilon) = [1 + \hat{\Phi}(\epsilon)]^{-1} \ \text{with} \ \hat{\Phi}(\epsilon) = \hat{k}_0^{-1}\hat{\phi}_1(\epsilon + \hat{\mathcal{L}}_2) \tag{4.6}$$

and

$$\hat{\phi}_1(\epsilon) = \langle\mathbf{z}^t_1|\hat{g}_1(\epsilon)|\mathbf{z}^t_1\rangle^{-1} = \langle\mathbf{z}^t_1|(\epsilon + \hat{\Lambda}_1)^{-1}|\mathbf{z}^t_1\rangle^{-1}. \tag{4.7}$$

The expressions (4.4)-(4.7) relates the evolution operator $\hat{\tilde{\mathbb{G}}}(\epsilon)$ of the total system in the combined space $\{x \otimes \mathbf{z}\}$ to that $\hat{g}_1(\epsilon) = (\epsilon + \hat{\Lambda}_1)^{-1}$ in the subspace $\{\mathbf{Z}_1\}$.

The procedure presented above allows one to express $\hat{\phi}_1(\epsilon + \mathcal{L}_2)$ in terms of the evolution operator $\hat{g}_2(\epsilon)$ in the reduced subspace $\{\mathbf{Z}_2\}$. The expressions similar to eqs. (4.4)- (4.7), as applied to the operator $\hat{g}^{tt}_1(\epsilon)$, yield

$$\hat{\phi}_1(\epsilon) = \langle\mathbf{z}^t_2|[\hat{\phi}_2(\epsilon + \hat{\mathcal{L}}_3) + \hat{\Lambda}_2]^{-1}|\mathbf{z}^t_2\rangle^{-1} \tag{4.8}$$

with

$$\hat{\phi}_2(\epsilon) = \hat{g}^{tt}_2(\epsilon)^{-1} \tag{4.9}$$

and

$$\hat{g}^{tt}_2(\epsilon) = \langle\mathbf{z}^t_2|\hat{g}_2(\epsilon)|\mathbf{z}^t_2\rangle = \langle\mathbf{z}^t_2|(\epsilon + \hat{\Lambda}_2)^{-1}|\mathbf{z}^t_2\rangle. \tag{4.10}$$

By continuing the proposed procedure one gets the expression for the function $\hat{\Phi}(\epsilon)$ in terms of $\hat{g}^{tt}_q(\epsilon)$ (with $q \ge 1$), and therefore in terms of functions

$$\hat{\phi}_q(\epsilon) = \langle\mathbf{z}^t_q|(\epsilon + \hat{\Lambda}_q)^{-1}|\mathbf{z}^t_q\rangle^{-1}, \tag{4.11}$$

which describe memory effects in the system as a result of Markovian motion in $\{z_q\}$-subspace, i.e. without controlling interaction with other subspaces $\{z_i\}$ with $i > q$:

$$\hat{\Phi}(\epsilon) = \hat{k}_0^{-1}\hat{\phi}_1(\hat{\phi}_2(\hat{\phi}_3(\ldots)))(\epsilon). \tag{4.12}$$

2. Examples

To illustrate the obtained results we will consider two limiting examples of two-state exponential and anomalously slow inverse power type controlling processes.

a. Two-state controlling processes. The exponential two-model for controlling processes corresponds to the simple expression for the evolution operator

$$\hat{\phi}_q(\epsilon) \sim \langle \mathbf{z}_q^t|(\epsilon+\hat{\Lambda}_q)^{-1}|\mathbf{z}_q^t\rangle^{-1} = \epsilon_q/\epsilon - \kappa_q/(\epsilon+\xi_q), \quad (4.13)$$

where ϵ_q, κ_q, and ξ_q are some constant parameters. Obviously, formula (4.12) with $\hat{\phi}_q(\epsilon)$ presented in eq. (4.13) gives the expression for $\hat{\Phi}(\epsilon)$ of type of continued fraction which predicts multiexponential behavior of the PDF matrix $\hat{w}(t)$.

b. Anomalously slow controlling processes. Another very important model of controlling processes describes anomalously slow inverse-power type behavior of the PDF $\hat{w}_1(t)$. This model is realized by taking a weak fractional-power type dependence of $\hat{\phi}_q(\epsilon)$ [7]:

$$\hat{\phi}_q(\epsilon) = w_q(\epsilon/w_q)^{\alpha_q}, \;\; \alpha_q < 1, \quad (4.14)$$

where $w_q = \zeta_q k_{0_q}$ and $\zeta_q \sim 1$. Substitution of eq. (4.14) into formula (4.12) yields

$$\hat{\Phi}(\epsilon) = R_0(\epsilon/w_N)^{\alpha_0}, \;\; \text{with } \alpha_0 = \prod\nolimits_{q=1}^{N}\alpha_q < 1 \quad (4.15)$$

and $R_0 = \prod_{q=1}^{N}(w_q/w_{q-1})^{\gamma_q}$, where $\gamma_q = \prod_{i=1}^{q}\alpha_i$.

Note that in the simplest case of identical rates w_q: $w_q = \bar{w}$, $(q \ge 1)$, when $R_0 = 1$ and $w_N \approx \bar{w}$, formula (4.15) is represented in a simple form $\hat{\Phi}(\epsilon) \approx (\epsilon/\bar{w})^{\alpha_0}$.

B. Fluctuating CTRW-jumps

1. General formulas

Another interesting problem in the CTRW theory, which can thoroughly be analyzed with the use of the proposed Markovian representation, concerns the kinetics of CTRW processes governed by stochastically fluctuating PDF matrices $\hat{\mathbb{W}}_\beta(t)$ of jumps (renewals) or corresponding matrices $\hat{w}_\beta(t)$ (here β is the number of jump).

The fact is that in conventional CTRW theories the PDF $\hat{W}_\beta(t)$ and the probability $\hat{P}_\beta(t)$ are assumed to be the same for all renewals $\beta > 1$ except the first one $(\beta = 0)$ and fixed functions of time [eq. (3.4)]. In this section we will discuss the extension of CTRW processes in which the fluctuation jump kinetics is controlled PDF and probability matrices whose functional form fluctuates leading, in particular, to the difference of $\hat{\mathbb{W}}_\beta(t)$ for different β.

The main difficulty in modeling fluctuating PDF matrices consists in necessity to take into account the normalization condition $\int_0^\infty dt\, W_\beta(t) = \widetilde{\hat{W}}_\beta(\epsilon = 0) = 1$, which ensures the population conservation in the process.

The Markovian representation, discussed in Sec. III, allows us to make the above problem tractable. Within this representation the fluctuations of PDF functions $\hat{w}_\beta(t)$ are assumed to be determined by the Markovian controlling process which is governed by the fluctuating operator $\hat{\Lambda}$. The representation offers quite natural and fairly simple way of description of fluctuating $\hat{\Lambda}(t)$ by suggesting this operator to depend on the parameter $\mathbf{y}(t)$ which undergoes stochastic Markovian fluctuations. In this model the validity of the normalization relation for $\hat{w}_\beta(t)$ can be ensured by taking proper form of the operator $\hat{\Lambda}(t)$. It is clear from eq. (3.9) that this relation will be fulfilled if during evolution in $\{z\}$-space, described by fluctuating $\hat{\Lambda}(t)$, the population is conserved, i.e. eq. (3.9) can be represented in the form $\dot{\varphi} = -(\nabla_z \cdot \mathbf{J}_z(\varphi, t))$, where $\mathbf{J}_z(\varphi, t)$ is the fluctuating flux in $\{z\}$-space. For instance, the population is certainly conserved in processes governed the Smoluchowski operator (3.11). More clearly this fact will be illustrated below with some examples.

In general in the Markovian model proposed above, the PDF $\sigma(y|t)$ representing $y(t)$-fluctuation process satisfies equation of type of (3.9):

$$\dot{\sigma} = -\hat{\Lambda}_y\sigma \quad \text{with} \quad \sigma(\mathbf{y}, 0) = \sigma_i(\mathbf{y}), \quad (4.16)$$

where $\hat{\Lambda}_y$ is the operator responsible for the evolution in $\{y\}$-space.

Similar to the case of $\hat{\mathcal{L}}(t)$ fluctuations caused by Markovian fluctuating $\mathbf{z}(t)$-parameter, the consideration of the effect of $\mathbf{y}(t)$-fluctuations reduces to analyzing the SLE for the evolution operator $\hat{\mathbb{G}}_{\mathbf{r}}(t)$ in the combined space $\{\mathbf{r}\} = \{x \otimes z \otimes y\}$:

$$\dot{\hat{\mathbb{G}}}_{\mathbf{r}} = -(\hat{\mathcal{L}} + \hat{\Lambda}_z + \hat{\Lambda}_y)\hat{\mathbb{G}}_{\mathbf{r}} \;\text{ with }\; \hat{\mathbb{G}}_{\mathbf{r}}(\mathbf{r}, \mathbf{r}_i|0) = \delta_{\mathbf{r}\mathbf{r}_i}. \quad (4.17)$$

so that

$$\widetilde{\hat{\mathbb{G}}} = \langle \mathbf{y}_e|\langle \mathbf{z}_e|(\epsilon + \hat{\mathcal{L}} + \hat{\Lambda}_y + \hat{\Lambda}_z)^{-1}|\mathbf{z}_i\rangle|\mathbf{y}_i\rangle, \quad (4.18)$$

where $|\mathbf{y}_i\rangle \equiv \sigma_i(\mathbf{y})$ is the initial $\mathbf{y}(t)$-state in bra/ket notation and

$$\langle \widetilde{\hat{U}} \rangle = \langle (\epsilon + \hat{\mathcal{L}} + \hat{\Lambda}_y + \hat{\Lambda}_z)^{-1} \rangle_{\mathbf{r}}, \quad (4.19)$$

In eq. (4.17) both the jump operator $\hat{\mathcal{L}}$ and the controlling operator Λ_z can, in principle, depend on variable y as a parameter: $\hat{\mathcal{L}} \equiv \hat{\mathcal{L}}_z^y$ and $\Lambda_z \equiv \Lambda_z^y$.

Just these dependencies of $\hat{\mathcal{L}}$ and Λ_z on y model the effect of $\mathbf{y}(t)$ evolution [i.e. fluctuations of PDF functions $\hat{w}_j(t)$] on the kinetics of processes under study. In general, both dependencies can strongly contribute to the effect, however in this work we will concentrate on the analysis of the effect of $\hat{\mathcal{L}}_z^y$-dependence only, assuming that Λ_z is independent of $\mathbf{y}$.

In the Markovian model of $\hat{w}_j(t)$-fluctuations the problem reduces to the analysis of the SLE (4.17) which is similar to eq. (3.12) considered above in detail. In general, eq. (4.17) can be solved only numerically. In this work we will restrict ourselves to discussing some most important specific features of manifestation of $\hat{w}_j(t)$-fluctuations in the kinetics of the process within the simple analytically solvable variant of the proposed model.

2. Localized rate of fluctuation jumps

The variant of the Markovian model which allows for analytical analysis of the problem can be considered as an extension of the approximation of highly localized jump rates discussed above. Here we assume high localization of fluctuation jump rates in both spaces $\{z\}$ and $\{y\}$, i.e. in the combined space $\{z \otimes y\}$. In this limit the expression for $\hat{\mathcal{L}}_z^y$-dependence is given by

$$\hat{\mathcal{L}}_z^y = (1 - \hat{P}_x)\hat{k}_0 \hat{P}_{yz}^{tt} \text{ with } P_{yz}^{tt} = |\mathbf{y}_t \mathbf{z}_t\rangle\langle \mathbf{y}_t \mathbf{z}_t|, \quad (4.20)$$

where $|\mathbf{y}_t \mathbf{z}_t\rangle = |\mathbf{y}_t\rangle|\mathbf{z}_t\rangle$ is the jump state in $\{z \otimes y\}$-space which, for simplicity, is suggested to remain unchanged after jumps.

In the model (4.20) $y(t)$-fluctuations are represented as a set of jumps (blips). The statistics of jumps is characterized by the PDF $w_y(t)$ of waiting times of jumps whose Laplace transform can be written as:

$$\hat{\tilde{w}}_y(\epsilon) = [1 + \hat{\Phi}_y(\epsilon)]^{-1}, \quad (4.21)$$

where

$$\hat{\Phi}_y(\epsilon) = [\hat{k}_0 \langle \mathbf{y}_t|(\epsilon + \hat{\Lambda}_y)^{-1}|\mathbf{y}_t\rangle]^{-1}. \quad (4.22)$$

It is easily seen that from mathematical point of view the proposed model is close that formulated in Sec. III. The difference consists only in a larger dimensionality of the space of the controlling process ($\{z \otimes y\}$-space instead of $\{z\}$-one in the model considered in Sec. III) and in the corresponding difference in the form of the operator Λ which describes the controlling process ($\Lambda_z + \Lambda_y$ instead of Λ_z).

This similarity enables us to use the results obtained above. It follows from these results that the model (4.20) predicts the CTRW expression for the evolution operator $\hat{\tilde{\mathbb{G}}}$ of type of eq. (3.15):

$$\hat{\tilde{\mathbb{G}}}(\epsilon) = \hat{\tilde{\mathbb{P}}}_i(\epsilon) + \hat{\tilde{\mathbb{P}}}_t(\epsilon)[1 - \hat{\widetilde{\mathbb{W}}}_t(\epsilon)]^{-1}\hat{\widetilde{\mathbb{W}}}_i(\epsilon), \quad (4.23)$$

where

$$\hat{\widetilde{\mathbb{W}}}_j(\epsilon) = \hat{P}_x \hat{\tilde{w}}_{f_j}(\epsilon) \text{ and } \hat{\tilde{\mathbb{P}}}_j(\epsilon) = \epsilon^{-1}[1 - \hat{\tilde{w}}_{f_j}(\epsilon)], \quad (4.24)$$

$(j = i, t)$, and

$$\hat{\tilde{w}}_{f_j}(\epsilon) = [1 + \hat{\Phi}_{f_j}(\epsilon)]^{-1}, \quad (4.25)$$

however, with matrices

$$\hat{\Phi}_{f_j}(\epsilon) = [\hat{k}_0 \langle \mathbf{y}_t \mathbf{z}_t|(\epsilon + \hat{\Lambda}_y + \hat{\Lambda}_z)^{-1}|\mathbf{y}_t \mathbf{z}_j\rangle]^{-1}. \quad (4.26)$$

whose behavior essentially depends on specific features of stochastic evolution of $\mathbf{y}(t)$.

This expression makes it possible to easily analyze some specific features of the effect of $\mathbf{y}(t)$-fluctuations. First of all, it is of special interest to discuss two natural limits: slow and fast $\mathbf{y}(t)$-fluctuations.

a. The limit of slow $y(t)$- and $\hat{w}_j(t)$-fluctuation. In the limit of slow fluctuations $y(t)$, when the characteristic time of evolution in $\{y\}$-space, τ_y, is larger than that of the PDFs $\hat{w}_j(t)$, $\tau_w : \tau_y \gg \tau_w$, one can neglect the term $\hat{\Lambda}_y$ in eqs. (4.17)-(4.19) thus arriving at the expression for $\hat{\mathbb{G}}$ and $\langle \hat{\tilde{U}} \rangle$ of type of those obtained in CTRW approach (see Sec. III) but averaged over y-distribution $|\sigma_i(\mathbf{y})\rangle \equiv |\mathbf{y}_i\rangle$. In the short range model considered in this limit one gets the expression (4.23)-(4.26) for $\hat{\tilde{\mathbb{G}}}(\epsilon)$ with:

$$\hat{\Phi}_{f_j}(\epsilon) \approx [\hat{k}_0 \langle \mathbf{y}_t|\mathbf{y}_j\rangle\langle \mathbf{z}_t|(\epsilon + \hat{\Lambda}_z)^{-1}|\mathbf{z}_j\rangle]^{-1}. \quad (4.27)$$

b. The limit of fast $y(t)$- and $\hat{w}_j(t)$-fluctuations. In the opposite limit of fast fluctuations $\mathbf{y}(t)$, corresponding to $\tau_y \ll \tau_w$, the SLE (4.17) still yields the CTRW-like formula for the evolution operator $\hat{\mathbb{G}}$. In this formula, however, the matrices $\hat{\widetilde{\mathbb{W}}}_j(\epsilon)$ and $\hat{\tilde{\mathbb{P}}}_j(\epsilon)$ are expressed in terms of the PDF matrix $\hat{w}_j(t)$ obtained by means of relations (3.17)-(3.21) which are evaluated with transition matrices $\hat{k}_z^d(\mathbf{y})$ ana $\hat{k}_z^n(\mathbf{y})$ [see eq. (3.14)] averaged over the equilibrium y-distribution $\sigma_e(\mathbf{y}) \equiv |\mathbf{y}_e\rangle$. In the fast fluctuation limit the short range model results in the expression (4.23)-(4.26) for $\hat{\tilde{\mathbb{G}}}(\epsilon)$ with:

$$\hat{\Phi}_{f_j}(\epsilon) \approx [\hat{k}_0 \langle \mathbf{y}_t|\mathbf{y}_e\rangle\langle \mathbf{z}_t|(\epsilon + \hat{\Lambda}_z)^{-1}|\mathbf{z}_j\rangle]^{-1}, \quad (4.28)$$

where $|\mathbf{y}_e\rangle$ is the equilibrium state in $\{y\}$-space.

It is seen from formulas (4.27) and (4.28) that in the considered simple model of localized jump rates the kinetics of the process is similar in both limits. The difference is only in the characteristic parameters. However, in general, in the case of delocalized jump rates, i.e. delocalized dependence of $\mathcal{L}_z^y$ on $\mathbf{y}$, the kinetics in these two limits is, of course, different though the asymptotic behavior at long times is expected to be similar. Below we will discuss the effect of delocalization of this dependence (see next Section).

In addition, strictly speaking, in the above analysis of two limits we have assumed that $\mathbf{y}(t)$-fluctuations are stationary which implies the existence of the equilibrium state $|\mathbf{y}_e\rangle$ of the operator Λ_y. It is of special interest to study the manifestation of non-stationary $\mathbf{z}(t)$-fluctuations which can be fairly strong especially in the case anomalously slow fluctuations [7].

3. Anomalous $y(t)$-fluctuations

Here we will analyze in detail the effect of anomalously slowly fluctuating coordinate $\mathbf{y}(t)$. In the Markovian representation this type of fluctuations can be described with the use of the free diffusion model for $y(t)$-evolution discussed in Sec. III.B1. In this model the operator Λ_y is given by

$$\hat{\Lambda}_y = -y^{1-n_y}\nabla_y(D_y y^{n_y-1}\nabla_y), \quad (4.29)$$

where $\nabla_y = \partial/\partial y$ and n_y is the $\{y\}$-space dimensionality (it is assumed that $n_y \leq 2$).

The model (4.29) predicts the anomalous long-time-tailed dependence of the characteristic PDF $w_y(t)$ [see formulas (4.21) and (4.22)]:

$$w_y(t) \sim 1/t^{1+n_y/2}. \tag{4.30}$$

These anomalous fluctuations of PDFs $w_\beta(t)$ of CTRW-jumps very strongly affect the kinetics of CTRW process. The effect can clearly be revealed by analyzing the behavior of the matrices $\Phi_{f_j}(\epsilon)$ [see eq. (4.26)]. Of special interest and importance is the asymptotic behavior of $\Phi_{f_j}(\epsilon)$ at $\epsilon \to 0$ which determines the long time asymptotic behavior of the process.

To demonstrate the specific features of the small-ϵ behavior of $\Phi_{f_j}(\epsilon)$ we will consider two examples of controlling processes in $\{z\}$-space: exponential (Poissonian) and anomalous long-time-tailed. Both these examples can properly be treated within the diffusion model.

a. Poissonian $\mathbf{z}(t)$ *controlling processes.* The Poissonian-like controlling processes can be described by the model of diffusive motion of the Brownian particle confined within the well in $\{z\}$-space. In this model at small ϵ, which correspond to times longer than the time τ_z of diffusive relaxation within the well $U(z)$ in $\{z\}$-space, the behavior of $\Phi_{f_i}(\epsilon)$ is determined by free diffusion in $\{y\}$-space and therefore

$$\Phi_{f_j}(\epsilon) \sim \epsilon^{n_y/2} \;\text{ and }\; w_{f_j}(t) \sim 1/t^{1+n_y/2}, \tag{4.31}$$

$(j = i, t)$. This result shows that originally normal CTRW processes become anomalous as a result of anomalous $w_j(t)$-fluctuations and the resultant behavior of $w_j(t) \equiv w_{f_j}(t)$ coincides with that of $y(t)$-fluctuations, i.e. anomalous $y(t)$-fluctuations strongly modify the kinetics of normal CTRW processes.

b. Long-time-tailed $\mathbf{z}(t)$ *controlling processes.* The long-time tailed case is represented by the free diffusion model in $\{z\}$-space $[U(z) = 0]$. In this model the small ϵ-behavior of $\Phi_{f_i}(\epsilon)$ is determined by free diffusion in the total $\{y \otimes z\}$-space of dimensionality $n_y + n_z$. This means that in the model of localized jump rates

$$\Phi_{f_j}(\epsilon) \sim \epsilon^{n_{yz}/2} \;\text{ and }\; w_{f_j}(t) \sim 1/t^{1+n_{yz}/2}, \tag{4.32}$$

where $n_{yz} = n_y + n_z$. Formula (4.32) demonstrates the strong effect of $w_j(t)$-fluctuations on the kinetics of the originally anomalous CTRW processes. Noteworthy is that $w_j(t)$-fluctuations result in the increase of the anomaly parameter α which determines the long time behavior of $w_j(t)$: $w_j(t) \sim 1/t^{1+\alpha}$.

It is interesting to note that the effect of fluctuations can lead to the crucial change of fluctuation jump statistics. The fact is that usually CTRW theories assume that $\int_0^\infty dt\, W_\beta(t) = 1$. This relation ensures conservation of normalization (or population) in CTRW processes. In the free diffusion approximation for the controlling process this conservation relation is fulfilled if the dimensionality of the space is small, when the statistics of reoccurrences in the jump state is recurrent. That is why we have assumed $n_y < 2$ and $n_z < 2$. These two inequalities, however, do not warrantee the same inequality for $n_{yz} = n_y + n_z$. In principle, one can get $n_{yz} > 2$ and in this case the the statistics of reoccurrences becomes transient which means that $\int_0^\infty dt\, w_j(t < 1$. Such CTRW processes with decay are known in the probability theory although are not applied widely [27].

V. DISCUSSION

The results obtained in this work have demonstrated that the Markovian representation is very useful for the analysis and extensions of the CTRW approach. In this section we will discuss some specific features of the proposed method and obtained results.

But first we would like to emphasize the important point concerning the relation of this representation to conventional approaches applied in the theory of CTRW processes. The fact is that the Markovian representation can be considered as a convenient method of realizing subordination (in a fairly general form) which is conventionally used as a basis for formulation of the CTRW approach [17, 27, 28]. Within this representation the subordination is associated with the controlling process in $\{\mathbf{z}\}$-space (Sec. III.B). The important advantage of the proposed realization consists in simplification of the description of non-Markovian kinetic problems by reducing the treatment to manipulations with linear operators. Moreover, in many cases the representation allows for deep understanding and modeling of real stochastic processes in clear physical terms.

A. Applicability of CTRW approach

The Markovian equation (3.12), applied to deriving the Markovian representation, is of course more general than the CTRW approach and therefore is quite suitable for the analysis of applicability conditions of this approach. Here we will present some comments on this point.

The problem reduces to analyzing the statistics of fluctuation jumps controlled by the stochastic process in $\{z\}$-space which is determined by the evolution operator $\hat{G}(\epsilon) = (\epsilon + \hat{k}_z^d + \hat{\Lambda})^{-1}$, where $\hat{k}_z^d$ is the jump-rate matrix diagonal in $\{x\}$- and $\{z\}$-spaces [see eq. (3.16)].

In the limit of high localization of jump-rate matrix $\hat{k}_z^d$ (in $\{z\}$-space) defined by eq. (3.14) the Markovian equation (3.12) leads to the evolution operator $\hat{\tilde{\mathbb{G}}}(\epsilon)$ corresponding to the CTRW approach. In the case of extended $\hat{k}_z^d$, however, the CTRW approach is not valid, strictly speaking.

To find the conditions which can ensure the applicability of the CTRW approach we will consider above-proposed simple and fairly flexible diffusion model for the controlling operator $\hat{\Lambda}$ [eq. (3.11)].

1) *Poissonian-like statistics.* Within the diffusion model the Poissonian-like jump statistics is described by the evolution operator $\hat{G}(\epsilon)$ with the Smoluchowski operator $\hat{\Lambda}$, in which the potential $u(z)$ is of type of infinitely deep potential well, say, of width a_u, i.e. $u(a_u) \sim 1$. At long times the proposed model predicts exponentially decreasing function $w_n(t)$ with the character time $\tau_n \sim a_u^2/D_z$.

It is easily seen that in this model the limit of high localization is realized for $a_k << a_u$, where a_k is the characteristic width of the function $\hat{k}_z^d$.

2) *Anomalous long-time tailed statistics.* The more interesting case of long-time tailed jump statistics is represented by the free diffusion variant of the diffusion model ($u(z) = 0$), which in the limit of highly localized $\hat{k}_z^d$ predicts the PDF $w_j(t) \sim 1/t^{1+n_z/2}$, where n_z is the dimensionality of $\{z\}$-space.

In the case of long-time tailed statistics the condition of applicability of the CTRW approach (applicability of the approximation of highly localized jump rates) is less trivial than that formulated above for Poissonian-like statistics. The fact is that in the absence of the potential one needs to compare the characteristic width of $\hat{k}_z^d$ in $\{z\}$-space with that of the PDF $\varphi(z,t)$ of the controlling process [see eq. (3.9)] which yield the condition $a_k^2 << D_z t$.

It is important to note that to satisfy this condition the function $\hat{k}_z^d$ does not need to be very short range. It can be shown that, for example, in the case $\hat{k}_{\mathbf{z}}^d \sim 1/z^m$ the proper parameter a_k can be introduced for $m > 3$ [29]. In addition in the case $m > 3$ the long time behavior of the PDF is shown to be identical to that for really short range $\hat{k}_z^d$ with the corresponding size a_k. This means that for $m > 3$ at long times $a_k << D_z t$ the CTRW approach is quite applicable.

This brief analysis shows that *in the large class of anomalous non-Markovian models the long time behavior of the evolution operator $\hat{\mathbb{G}}(t)$ is correctly described by the CTRW approach.*

B. Extended CTRW approaches

In Sec. IV we have studied most general features of two extensions: CTRW with cascaded controlling processes and CTRW processes with fluctuating PDFs. Below we will discuss in detail some particular predictions of the extended CTRW-approaches.

1. Cascaded controlling processes

The model of cascaded control is in reality a good tool for the analysis of the kinetic process in fractal structures. The cascade of coupled processes can properly model the kinetic coupling of structures of different size. The model is very useful for the analysis of kinetics of processes in selfsimilar and complex structures [13].

In particular, let us discuss anomalous relaxation in highly disordered structures. In such structures the controlling processes are often quite adequately described by the anomalous model considered in Sec. IV.B in which the controlling evolution functions $\hat{\phi}_q(\epsilon) \sim \epsilon^{\alpha_q}$ with $\alpha_q < 1$ ($q = 1, \ldots, N$) [see eq. (4.14)] [7]. In this model formula (4.15) predicts interesting behavior of the total controlling evolution function $\hat{\Phi}(\epsilon) \sim \epsilon^{\alpha_0}$, where $\alpha_0 = \prod_{i=1}^N \alpha_i$. In principle, the product in equation for α_0 is convergent for properly behaving α_i as a function of i. Corresponding criteria are known [30] but it is evident that the necessary condition is $\alpha_{i\to\infty} \to 1$. This condition has fairly clear physical interpretation: if we escribe the controlling functions with larger numbers i to the structures of smaller size, then approaching of α_i to unity results from the evident fact that in structures of smaller size the anomalous effects, caused by disorder of medium, are expected to be weaker. Weakness of effects manifests itself in the reduction of processes to Markovian with the increase of i, i.e. just in the relation $\alpha_{i\to\infty} \to 1$ or $\alpha_i = 1$ at i larger than some characteristic number N.

It is also important to note another interesting prediction of the expression (4.15). It shows that in the case of cascaded controlling process the anomaly of the processes in the cascade is accumulated. This effect manifests itself in the decrease of α_0 with the increase of the number N of coupled processes. In particular, for a large number N of weakly anomalous cascaded controlling processes with $\delta_i = 1 - \alpha_i \ll 1$ one gets the value

$$\alpha_0 \approx e^{-\sum_{j=1}^N \delta_j} \approx e^{-\int_0^N dj\,(1-\alpha_j)}, \tag{5.1}$$

which can be fairly small, corresponding to strongly anomalous process. Formula (5.1) can shed light on the mechanism of formation of relaxation anomaly in disordered systems.

2. Fluctuations of waiting time PDF matrices

The results obtained in Sec. IV.B demonstrate that fluctuations of waiting time PDF matrices can strongly manifest themselves in the kinetics of CTRW-like processes. It is, however, still worth to point out some specific features of this effect.

1) In our consideration in Sec. IV.B these fluctuations are assumed to result from those of the jump rate $\hat{k}_0$, i.e. from the dependence $\hat{k}_0(\mathbf{y})$. For simplicity, we have used the model of highly localized $\hat{k}_0(\mathbf{y}) = \hat{\kappa}_0|\mathbf{y}_t\rangle\langle\mathbf{y}_t|$. However, in accordance with conditions of applicability of CTRW approaches discussed above in Sec. V.A, the main conclusions on the effect of fluctuation in the kinetics of the process remain valid for delocalized dependencies $\hat{k}_0(\mathbf{y})$ as well, if these dependencies are sharp enough (for details see Sec. IV.B). In other words the obtained results are valid for the wide class of models based on the Markovian representation.

2) For the sake of simplicity of the analysis, the operators $\hat{\Lambda}_z$ and $\hat{\Lambda}_y$ have been suggested to be independent of y and z variables, respectively. Nevertheless, the major part of conclusions of Sec. IV.B are valid in the case of coupled processes in $\{y\}$- and $\{z\}$-spaces (i.e. for $[\hat{\Lambda}_z, \hat{\Lambda}_y] \neq 0$). The most general formulation in this case consists in replacement of the sum $\hat{\Lambda}_z + \hat{\Lambda}_y$ by some operator $\hat{\Lambda}_{yz}$ in the combined space $\{y \otimes z\}$.

3) To illustrate possible modifications of the fluctuation mechanism within the above general formulation we will briefly discuss the simple model which describes fluctuations occurring at moments of jump transitions. In principle, this model corresponds to the highly localized z-dependence of the operator $\hat{\Lambda}_y(\mathbf{z})$: $\hat{\Lambda}_y(\mathbf{z}) \sim \hat{P}_z^{tt}$, but in the end the dependence reduces to the following modification of the operator $\hat{\mathcal{L}}$ [see eqs. (3.7) and (3.14)]

$$\hat{\mathcal{L}} = \hat{k}_0(\mathbf{y})\hat{P}_z^{tt} - \hat{P}_x\hat{P}_y\hat{k}_0(\mathbf{y})\hat{P}_z^{nt}. \tag{5.2}$$

Here $\hat{P}_y$ describes sudden change of $\mathbf{y}$-coordinate simultaneously with the jump in $\{x\}$-space. Of course $\hat{P}_y$ satisfies the normalization condition $\langle \mathbf{y}_e|(1-\hat{P}_y) = 0$, where $\langle \mathbf{y}_e| = \sum_i \langle \mathbf{y}_j|$ is the adjoint equilibrium vector in $\{y\}$-space. For example, in the simplest variant of sudden relaxation in $\{y\}$-space $\hat{P}_y = |\mathbf{y}_e\rangle\langle \mathbf{y}_e|$.

Substitution of the expression (5.2) into the SLE (3.12) and subsequent manipulations similar to those presented in Sec. III yield

$$\hat{\widetilde{\mathbb{G}}}(\epsilon) = \langle \mathbf{y}_e|\hat{\widetilde{\mathbb{G}}}_y(\epsilon)|\mathbf{y}_i\rangle, \tag{5.3}$$

where

$$\hat{\widetilde{\mathbb{G}}}_y(\epsilon) = \hat{\widetilde{\mathbb{P}}}_{y_i}(\epsilon) + \hat{\widetilde{\mathbb{P}}}_{y_n}(\epsilon)[1 - \widetilde{\mathbb{W}}_{y_n}(\epsilon)]^{-1}\hat{\widetilde{\mathbb{W}}}_{y_i}(\epsilon). \tag{5.4}$$

The parameters in this formula are similar to those defined in eqs. (3.15)-(3.21). The only difference consists in additional term $\hat{P}_y$ in the expression for $\hat{\widetilde{\mathbb{W}}}_{y_j}, (j = i, n)$, and in y-dependence of the parameters resulting from the dependence $\hat{k}_0(\mathbf{y})$. According to eq. (3.17), $\hat{\widetilde{\mathbb{W}}}_{y_j} = \hat{P}_x\hat{P}_y\hat{\widetilde{w}}_{y_j}(\epsilon)$, where $\widetilde{w}_{y_j}(\epsilon) = \hat{k}_0(y)\langle \mathbf{z}_t|\hat{G}_y(\epsilon)|\mathbf{z}_j\rangle$ and $\hat{G}_y(\epsilon)$ is obtained with $\hat{k}_0(\mathbf{y})$. This means that formula (5.4) coincides with eq. (3.17) in which $\hat{P}_x$ replaced by $\hat{P}_{xy} = \hat{P}_x\hat{P}_y$. In other words this formula describes jump-like migration in $\{x \otimes y\}$-space though with jump rate $\hat{k}_0(y)$ which has addition dependence on y. Moreover, the expression (5.4) can also be represented in the form (4.5) suitable for the analysis of the limit of diffusion-like motion governed by the operator $\hat{\mathcal{L}}_{xy} = (1 - \hat{P}_{xy})\hat{k}_0(\mathbf{y})$ describing diffusion with the coefficient $\hat{D}(\mathbf{y}) \sim \hat{k}_0(\mathbf{y})$ in some effective potential [31].

It is important to note that in this model (unlike the general model discussed in Sec. IV.B) $\hat{\mathbb{W}}(t)$-fluctuations, caused by stochastic motion in $\{y\}$-space, do not lead to the significant and universal change of the long time behavior of matrices $\hat{w}_{f_j}(t)$ defined in eq. (4.24) [see eqs. (4.31) and (4.32)]. Some change of $\hat{w}_{f_j}(t)$-behavior is, in principle, possible due to the effect of $\hat{D}(\mathbf{y}) \sim \hat{k}_0(\mathbf{y})$-dependence, but this change is not universal, strongly depending on specific features of $\hat{k}_0(\mathbf{y})$-behavior.

It is also worth noting that the model, which has something in common with the considered particular variant of our general model, is recently discussed in ref. [22]. Restricting ourselves to brief comments we would only like to emphasize that, as the analysis of above simple variant shows, the general method developed in our work enables one to represent the results obtained in this paper in very compact and general form.

4) From very beginning in our consideration the extensions have been discussed within the Markovian representation reducing the problem to the analysis of multidimensional SLE. In so doing we have not interpreted the results in terms of the conventional approach appealing to stochastic properties of fluctuating PDFs of consecutive jumps $\hat{\mathbb{W}}_\beta(t)$ (see Sec. III). It is clear that the case of most pronounced effect of fluctuations corresponds to strong long time tailed correlations of the PDFs and of course such expression can be found without difficulties. For example, one can consider the model assuming the process of stochastic change of $\hat{\mathbb{W}}_\beta(t)$ localized (in time) near the time of fluctuation jump. In this model the stochastic change of PDFs can be described by introducing additional fluctuation matrix $\hat{\mathbb{W}}_c(t)$ with which conventional convolution terms of type $\int_0^{t_2} dt_1\hat{\mathbb{W}}(t-t_1)\hat{\mathbb{W}}(t_1-t_0)$ are transformed into $\int_0^{t_2} dt_1\hat{\mathbb{W}}(t-t_1)\int_0^{t_1} d\tau\hat{\mathbb{W}}_c(t_1-\tau)\hat{\mathbb{W}}(\tau-t_0)$. It is clear that in the limit of very short range dependencies $\hat{\mathbb{W}}_c(t)$ this model reduces to that considered above. However, even in the most general formulation it can be treated as a particular variant of the above-proposed approach based on the Markovian representation.

The analysis of possible other variants of extensions which can be analyzed within the Markovian representation will be presented elsewhere.

VI. CONCLUSIONS

This work concerns detailed discussion and applications of the Markovian representation of non-Markovian CTRW-like processes and, in particular, non-Markovian CTRW-based SLE. In reality, however the Markovian representation, reducing the problem to the study of the multidimensional Markovian SLE, is more general than the CTRW approach and allows one not only to analyze the applicability of this approach but also develop some extensions. In our work we have considered two of them describing the effect of cascaded controlling processes and fluctuations of jump PDFs. The number of extensions is, however, very large. For example, interesting effects can be predicted in variants of CTRW approaches combining two above-mentioned extensions. Further generalizations and applications is a subject of subsequent publications.

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[1] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, New York, 1985).
[2] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Fluctuations* (W. A. Benjamin, Inc., London, 1975).
[3] A. Abragam, *The principles of nuclear magnetism* (Clarendon Press, Oxford, 1961).
[4] J. W. Haus and K. W. Kehr, Phys. Rep. **150**, 263 (1987).
[5] R. Kubo, J. Math. Phys. **4**, 174 (1963).
[6] J.-B. Bouchaud and A. Georges, Phys. Rep. **195**, 12 (1990).
[7] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[8] G. Margolin and E. Barkai, Phys. Rev. Lett. **94**, 080601 (2005).
[9] G. Bel and E. Barkai, Phys. Rev. Lett. **94**, 240602 (2005); Phys. Rev. **E73**, 016125 (2006).
[10] P. Allegrini, P. Grigolini, L. Palatella, and B. J. West, Phys. Rev. **E70**, 046118 (2004).
[11] P. Allegrini, G. Aquino, P. Grigolini, L. Palatella, A. Rosa, and B. J. West, Phys. Rev. **E71**, 066109 (2005).
[12] E. Barkai, e-print cond-mat/0608155.
[13] B. West and W. Deering, Phys. Rep. **246**, 1 (1994).
[14] P. Grigolini, in *Metastability and Nonextensivity*, edited by C. Beck, G. Benedek, A. Rapisadra, and C. Tsallis (World Scientific, Singapore, 2005).
[15] F. Barbi, M. Bologna, and P. Grigolini, Phys. Rev. Lett. **95**, 220601 (2005).
[16] H. Scher and E. W. Montroll, Phys. Rev. **B12**, 2455 (1975).
[17] E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).
[18] B. J. West, M. Bologna,and P. Grigolini, *Physics of Fractal Operators* (Springer-Verlag, New York, 2002).
[19] O. Flomenborn,K. Velonia, D. Loos, S. Masuo, Mircea Cotlet, Y. EngelBorghs, A. E. Rowan, R. J. M. Nolte, M. Van der Auweraer, F. C. Schryver, and J. Klafter, PNAS **102**, 2368 (2005).
[20] A. I. Shushin, Phys. Rev. **E67**, 061107 (2003).
[21] A. V. Chechkin, R. Gorenflo, and I. M. Sokolov, J. Phys. **A38**, L679 (2005).
[22] P. Allegrini, F. Barbi, P. Grigolini, and P. Paradisi, Phys. Rev. **E73**, 046136 (2006).
[23] A. I. Shushin and V. P. Sakun, Physica **A340**, 283 (2004).
[24] A. I. Shushin, New J. Phys. **7**, 21 (2005).
[25] K. Seki, M. Wojcik, and M. Tachiya, J. Chem. Phys. **119**, 2165 (2003).
[26] A. A. Zharikov, S. I. Temkin, and A. I. Burshtein, Chem. Phys. **103**, 1 (1986).
[27] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1971).
[28] I. M. Sokolov, Phys. Rev. **E63**, 011104 (2000).
[29] K. Seki, A. I. Shushin, M. Wojcik, and M. Tachiya, J. Phys. **C19**, 065117 (2007).
[30] I. S. Gradsteyn and I. M. Ryzhik, *Tables of Integrals, Serieses and Products* (Academic, San Diego, 1980).
[31] A. I. Shushin, J. Chem. Phys. **122**, 154504 (2005).